MTH 311 Advanced Calculus  
Homework Assignment #1  
Due in class on Monday, October 13, 2003

Directions: Solve the following problems. Each solution must neatly written and be mathematically and grammatically correct. Write up your solutions in a style similar to that used in our text when the author is presenting a proof. Sloppy written work or work that is neatly written but is not mathematically or grammatically well organized with not be graded.

1. Let $a$ and $b$ be real numbers.

   (a) Use mathematical induction to prove: $0 \leq a < b \Rightarrow a^n < b^n$ for all positive integers $n$.

   (b) Use proof by contradiction to prove: $0 \leq a < b \Rightarrow a^{1/n} < b^{1/n}$ for all positive integers $n$. (You may assume that $a^{1/n} = \sqrt[n]{a}$ is defined and has the usual meaning.)

   Remark: By (a) the function $x^n$ is increasing on $[0, \infty)$. By (b) the function $x^{1/n}$ is increasing on $[0, \infty)$.

2. (Bernoulli Inequalities) Use mathematical induction to prove (a). Then deduce (b) from (a) by first taking $n$th roots of the result in (a) and then by making an appropriate change of variable. (You may assume that each positive number $b$ has a positive $n$th root $b^{1/n}$.) Let $n$ be a positive integer and $u > -1$. Then

   (a) $(1 + u)^n \geq 1 + nu$

   (b) $(1 + u)^{1/n} \leq 1 + \frac{1}{n}u$

3. (On $n$th roots)

   (a) Let $a > 1$ and $n$ be a positive integer. Prove that $1 \leq a^{1/n} \leq 1 + \frac{1}{n} (a - 1)$.

   (b) What limit law from calculus enables you to conclude from (a) that $\lim_{n \to \infty} a^{1/n} = 1$?

   (c) If $0 < a < 1$ find $\lim_{n \to \infty} a^{1/n}$ and justify your answer.
4. Let \( n \) be a positive integer. Prove that \( n^3 < 2^n \) for all large \( n \). (This language means there exists an positive integer \( n_0 \) such that \( n^3 < 2^n \) for all \( n \geq n_0 \).) As part of this problem find the smallest \( n_0 \) that works.

5. Assume the product rule for differentiation that you learned in first-term calculus. Prove Leibniz Rule for differentiating a product of two functions: For any positive integer \( n \),

\[
(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}
\]

Before you can prove anything you will have to decide what must be assumed about the functions \( u = u(x) \) and \( v = v(x) \). Here the superscript \((k)\) means differentiation

\[
u^{(k)} = \frac{d^k u}{dx^k}
\]