Infinite Sequences of Real Numbers
(AKA ordered lists)

DEF. An **infinite sequence** of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$.

Usually (infinite) sequences are written as lists, such as,

$$\{x_n\}_{n=1}^{\infty}, \{x_n\}, \ x_1, x_2, x_3, \ldots$$

where $x_n = f(n)$.

DEF. An infinite sequence $\{x_n\}_{n=1}^{\infty}$ has the real number $a$ as a **limit** if given an real number $\varepsilon > 0$ there is a corresponding $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |x_n - a| < \varepsilon$$

in which case we write

$$\lim_{n \to \infty} x_n = a, \text{ or } x_n \to a \text{ as } n \to \infty.$$  

DEF. If a sequence has a **finite** limit we say it **converges**.

Facts:

The sequence $\{1/n\}$ converges and has limit 0.

A sequence may not have a limit.
Limits are unique if they exist.
Explicit Calculations With Limits

The process of taking a limit interacts as you would expect with the basic operations +, −, ×, and ÷ and with the order properties of real numbers.

**Th.** Suppose \( \{x_n\} \) and \( \{y_n\} \) are convergent sequences in \( \mathbb{R} \) and \( c \in \mathbb{R} \). Then the limits on the left all exist and have the indicated value:

\[
\lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n \\
\lim_{n \to \infty} (cx_n) = c \lim_{n \to \infty} x_n \\
\lim_{n \to \infty} (x_n y_n) = \lim_{n \to \infty} x_n \cdot \lim_{n \to \infty} y_n
\]

If, in addition, \( \lim_{n \to \infty} y_n \neq 0 \), then

\[
\lim_{n \to \infty} \left( \frac{x_n}{y_n} \right) = \frac{\lim_{n \to \infty} x_n}{\lim_{n \to \infty} y_n}
\]

**Th.** Suppose \( \{x_n\} \) and \( \{y_n\} \) are convergent sequences in \( \mathbb{R} \) and

\( x_n < y_n \) for all large \( n \)

or

\( x_n \leq y_n \) for all large \( n \)

Then

\[
\lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n
\]
The Squeeze Law  
(AKA The Three Sequences Theorem)

**Th.** Suppose \( \{x_n\} \), \( \{y_n\} \), and \( \{z_n\} \) are sequences in \( \mathbb{R} \) with 
\[
x_n \leq y_n \leq z_n \text{ for all large } n
\]
\[
\exists \lim_{n \to \infty} x_n \text{ and } \exists \lim_{n \to \infty} z_n \text{ and } \lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n.
\]
Then 
\[
\exists \lim_{n \to \infty} y_n \text{ and all three limits are equal.}
\]

**Corollaries:**

1. \[ |y_n| \leq z_n \rightarrow 0 \Rightarrow \exists \lim_{n \to \infty} y_n = 0. \]

2. If \( \{x_n\} \) is bounded (which means \( \exists M > 0 \) such that \( |x_n| \leq M \) for all \( n \)) and if \( y_n \rightarrow 0 \) as \( n \rightarrow \infty \), then \( \exists \lim_{n \to \infty} x_n y_n = 0. \)
The Dilemma With Limits!

In significant applications in which limits occur you generate a sequence \( \{x_n\} \) whose terms get "closer and closer" to the solution of a difficult practical or theoretical problem that you cannot solve by more elementary means. Your hope is that "closer and closer" means the sequence converges and that its limit, say \( a \), is the solution to your problem.

Here is the dilemma: You don’t know \( a \). If you did, you wouldn’t need the sequence! If you don’t know \( a \), then you can’t use the definition of a limit to check that the sequence has limit \( a \).

What are you to do?

You need ways to guarantee that a sequence converges without knowing its limit in advance! Stay turned.
Basic Properties a Sequence May Have

DEF. A sequence \( \{x_n\} \) of real numbers is **bounded above** if \( \exists M \in \mathbb{R} \) such that \( x_n \leq M \) for all \( n \);

**bounded below** if \( \exists m \in \mathbb{R} \) such that \( x_n \geq m \) for all \( n \);

**bounded** if it is both bounded above and bounded below.

Facts:
1. \( \{x_n\} \) is bounded \( \iff \exists M \in \mathbb{R} \) such that \( |x_n| \leq M \) for all \( n \).

2. Convergent sequences are bounded.

3. Bounded sequences need not converge.

4. Bounded sequences often contain convergent subsequences.

DEF. A **subsequence** of a sequence \( \{x_n\}_{n=1}^{\infty} \) is a sequence of the form

\[
\{x_{n_k}\}_{k=1}^{\infty} = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\}
\]
where $1 \leq n_1 < n_2 < n_3 < \cdots$
Increasing and Decreasing Sequences

DEF. A sequence \( \{x_n\} \) of real numbers is
- **increasing** if \( x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots \);
- **decreasing** if \( x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq \cdots \);
- **strictly increasing** if \( x_1 < x_2 < x_3 < \cdots < x_n < \cdots \);
- **strictly decreasing** if \( x_1 > x_2 > x_3 > \cdots > x_n > \cdots \);
- **monotone** if it is either increasing or decreasing.

Monotone sequences are important because they give one way out of our dilemma.

**Th. [Monotone Convergence Theorem]** Every *bounded monotone* sequence of real numbers converges.

**Th. [MCT – full disclosure version]**
1. An increasing sequence that is bounded above converges (to its least upper bound).
2. A decreasing sequence that is bounded below converges (to its greatest lower bound).
3. An increasing sequence that is not bounded above diverges to \( +\infty \).
4. A decreasing sequence that is not bounded below diverges to $-\infty$. 
Very Important Consequences of Monotonicity

**Cantor’s Nested Intervals Theorem:** If \( \{I_n\}_{n \in \mathbb{N}} \) is a sequence of *nonempty, closed, bounded, nested* intervals (nested means \( I_1 \supset I_2 \supset I_3 \supset \cdots \)), then

\[
\cap_{n \in \mathbb{N}} I_n \neq \emptyset
\]

Furthermore, if \( |I_n| = \text{length of } I_n \to 0 \) as \( n \to \infty \), then

\[
\cap_{n \in \mathbb{N}} I_n \text{ is a single real number.}
\]

**Th.** Every sequence contains a monotone subsequence.

**The Bolzano-Weierstrass Theorem:** Every bounded sequence of real numbers has (contains) a convergent subsequence.
Cauchy Criterion for Convergence

DEF. A sequence \( \{x_n\}_{n=1}^{\infty} \) is a **Cauchy sequence** if given any \( \varepsilon > 0 \) \( \exists N \) such that
\[
n, m > N \implies |x_n - x_m| < \varepsilon.
\]

Facts:

1. Convergent sequences are Cauchy.

2. Cauchy sequences are bounded.

3. (Theorem with a capital T): A sequence of real numbers converges if and only if it is a Cauchy sequence.

This theorem provides another way out of the limit dilemma.