Number Systems

\( \mathbb{N} = \{1, 2, 3, \ldots\} \) the positive integers
\( \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\} \) the integers
\( \mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ with } q \neq 0 \right\} \) the rational numbers
\( \mathbb{R} = \{ \text{numbers expressible by finite or unending decimal expansions} \} \)

+ makes sense in \( \mathbb{N} \)
+ and – make sense in \( \mathbb{Z} \)
+ and – and \( \times \) and \( \div \) make sense in \( \mathbb{Q} \) and \( \mathbb{R} \)
< makes sense in \( \mathbb{N} \) and \( \mathbb{Z} \) and \( \mathbb{Q} \) and \( \mathbb{R} \)

All the usual rules of arithmetic and inequalities hold in \( \mathbb{Q} \) and \( \mathbb{R} \).

Some but not all hold in \( \mathbb{N} \) and \( \mathbb{Z} \).

Postulate 1 [Field Axioms] in WRW ⇒ the usual arithmetic rules

Postulate 2 [Order Axioms] in WRW ⇒ the usual rules for inequalities
Absolute Value of a Real Number

\[ |a| = \begin{cases} 
  a & \text{if } a \geq 0 \\
  -a & \text{if } a \leq 0
\end{cases} \]

= dist(0, a) on the number line

**Basic Properties:** For all \( a, b \in \mathbb{R} \)

\[ |a| \geq 0 \text{ with } = \iff a = 0 \]

\[ |a - b| = |b - a| \]

\[ |a + b| \leq |a| + |b| \quad \text{The triangle inequality} \]

The triangle inequality implies

\[ ||a| - |b|| \leq |a - b| \]

\[ |a - b| = \text{dist}(a, b) \text{ on the number line} \]

Finally,

\[ |a| \leq M \iff -M \leq a \leq M \]
What is a Mathematical Theory?  
(such as calculus or geometry) 
A logical development of a subject that starts 
from basic assumptions (often called axioms and 
postulates) that are taken as self-evident. 
Thereafter, the only statements accepted as true 
or false are those that have been so proven 
reasoning from the axioms and using accepted 
rules of logical argumentation. 
The logical consequences of the axioms are 
called Theorems, Propositions, Lemmas, ... 
(according to a somewhat arbitrary scheme). 
The logical argumentation leading to the 
theorems, etc., is called "proof". 

Why is a Theory Needed? 
Some mathematical statements that appear true 
on geometric grounds or on physical grounds or 
according to your intuition turn out to be false!
Some Plausible Statements  
That Are Not True  

The rational numbers fill up the number line.
The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ converges.
The function $\ln x$ has a horizontal asymptote:
Three Main Types of Proof

1. **Mathematical Induction:**
   A basic property of the positive integers and far more powerful than first meets the eye.

2. **Direct Proof:**
   To prove If H, then C, you assume that H is true and use accepted principles of logical argumentation using the axioms and already established facts to deduce that C must logically follow.

3. **Indirect Proof (AKA Proof by Contradiction):**
   To prove If H, then C, you start by assuming H and that C is FALSE. Then you use accepted principles of logical argumentation to reach a contradiction. This proves that H implies C.

   A **contradiction** is a statement that is know to be false. (The statement that $x = 5$ and $x > 5$ is a contradiction.)

   Why is indirect proof a valid argument?
Postulate 3 [The Well-Ordering Axiom]. Every nonempty subset of positive integers has a smallest element.

Theorem [Principle of Mathematical Induction]: For each $n \in \mathbb{N}$, let $A(n)$ be a mathematical statement (an assertion that is either true or false). If

(a) $A(1)$ is true and
(b) for each $k \in \mathbb{N}$ for which $A(k)$ is true, $A(k + 1)$ also is true,

then $A(n)$ is true for all $n \in \mathbb{N}$.

In international intrigue the PMI is known as the domino theory.
Pascal’s Triangle

\[(a + b)^0 = 1\]
\[(a + b)^1 = a + b\]
\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]
\[(a + b)^4 = a^4 + 4a^3b^2 + 6a^2b^2 + 4ab^3 + b^4\]
\[\cdots\]

Pascal’s Triangle:

\begin{array}{cccccc}
\text{row 0} & 1 \\
\text{row 1} & 1 & 1 \\
\text{row 2} & 1 & 2 & 1 \\
\text{row 3} & 1 & 3 & 3 & 1 \\
\text{row 4} & 1 & 4 & 6 & 4 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}

The entries in Pascal’s triangle are called \textbf{binomial coefficients}.
The binomial coefficient in Pascal’s triangle in row \( n \) and position \( k \) from the left is denoted by

\[
\binom{n}{k}
\]
for \( k = 0, 1, 2, \ldots, n \) and \( n = 0, 1, 2, 3, \ldots \).

The binomial coefficients are defined inductively by (the pattern in Pascal’s triangle)

\[
\binom{n}{0} = 1, \quad \binom{n}{n} = 1
\]
\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad 1 \leq k \leq n - 1 \quad (*)
\]

If we define

\[
\binom{n}{-1} = 0, \quad \binom{n}{n+1} = 0
\]

(*) holds for \( k = 0, 1, 2, \ldots, n \).
Binomial Theorem  
(AKA Binomial Expansion)  
(FOIL at its best)

Theorem: If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

You should be able to guess the following rule by adding at most two rows to Pascal’s triangle and doing some trial and error calculations:

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{1(2)(3) \cdots (k)}$$

with $k$ factors in both the numerator and denominator. More compactly

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for $k = 0, 1, 2, \ldots, n$. 

Upper Bounds, Lower Bounds
and Related Matters

Let $E \subset \mathbb{R}$. A real number $M$ is an upper bound [lower bound] of $E$ if $x \leq M$ [$x \geq M$] for each $x \in E$.

A set $E \subset \mathbb{R}$ is bounded above [below] if it has an upper [lower] bound.

A set is bounded if it is both bounded above and bounded below.

A real number $M$ is the maximum (element) of a set $E$ if $M \in E$ and $M \geq x$ for every $x \in E$.
Notation: $M = \max E$.

A real number $m$ is the minimum (element) of a set $E$ if $m \in E$ and $m \leq x$ for every $x \in E$.
Notation: $m = \min E$.

If $\max E$ exists it is the least upper bound of the set.

If $\min E$ exists it is the greatest lower bound of the set.
What if there is no max or min?

A real number \( s \) is called the **supremum of** \( E \) or the **least upper bound of** \( E \) if \( s \) is an upper bound for \( E \) and \( s \leq M \) for any upper bound \( M \) of \( E \).

A real number \( t \) is called the **infimum of** \( E \) or the **greatest lower bound of** \( E \) if \( t \) is a lower bound for \( E \) and \( t \geq M \) for any lower bound \( M \) of \( E \).

Notation:

- If a set \( E \) has a supremum it is denoted by \( \text{sup} E \).
- If a set \( E \) has an infimum it is denoted by \( \text{inf} E \).

**Basic Facts:**

- If a set has an upper bound it has infinitely many.
- If a set has a supremum [or maximum], it is unique.
- If a set has a supremum, it may or it may not belong to the set.
  - If \( \text{sup} E \in E \), then \( \text{sup} E = \text{max} E \).

What are the corresponding statements for infima?
The Key to Working With Suprema

**Theorem** [*Approximation Property for Suprema*]. If a set $E$ has a supremum, $\sup E$, then for each $\varepsilon > 0$ there is an element $x$ in $E$ such that

$$\sup E - \varepsilon < x \leq \sup E$$

Equivalently, if a set $E$ has a supremum, $\sup E$, then for each $c < \sup E$ there is an element $x$ in $E$ such that

$$c < x \leq \sup E$$

What are the corresponding statements for infima?
The Completeness Axiom
(Distinguishing $\mathbb{R}$ from $\mathbb{Q}$ – there are no holes in the real number line.)

Postulate 4 [The completeness Axiom]: Every nonempty set of real numbers that is bounded above has a supremum.

Important Matters:

Every nonempty set $E$ that is bounded above has a supremum. However, the supremum need not belong to the set.

If a set has a supremum and that supremum belongs to the set, we also call the supremum the maximum (element) of the set.

Roughly speaking the supremum is a "substitute" for a maximum element of a set when one does not exist.

What are the corresponding statements for infima?
Uniqueness of the Real Number System

Postulates 1, 2, and 4 characterize the real numbers (up to isomorphism).
Fundamental Consequences of Completeness of $\mathbb{R}$

I. The Archimedian Property of $\mathbb{R}$.

There are arbitrarily large positive integers; equivalently, there are arbitrarily small reciprocals of positive integers. Many basic limit arguments rely on these facts.

**Theorem.** The following three equivalent properties hold in the real number system.

(a) $\mathbb{N}$ is not bounded above; that is, given any real number $b$ there is $n \in \mathbb{N}$ such that $n > b$.

(b) If $b \in \mathbb{R}$ and $x > 0$, then there is an $n \in \mathbb{N}$ such that $nx > b$.

(c) For each $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$. 
II. On Expected Properties of $\mathbb{Z}$ and $\mathbb{R}$.

Expected maxima and minima are there. The integers separate the reals as expected: $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n + 1)$

**Theorem.** If $E \subset \mathbb{Z}$ is nonempty and bounded above, then $\max E$ exists.

What is the corresponding result for $E \subset \mathbb{Z}$ bounded below? (It implies that the completeness axiom implies the well-ordering axiom for $\mathbb{N}$.)

**Theorem.** If $x \in \mathbb{R}$ there is a unique $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.

**Remark.** This $n$ is the **greatest integer in** $x$.  
**Notation:** $n = [x]$
III. $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Real numbers can be approximated arbitrarily closely by rational numbers.

**Theorem.** The following equivalent properties hold in the real number system:

(a) Given any two real numbers $a, b$ with $a < b$ there is a rational number $q$ such that $a < q < b$.

(b) Given $x \in \mathbb{R}$ and $\varepsilon > 0$ there is a $q \in \mathbb{Q}$ such that $|x - q| < \varepsilon$.

(c) Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$ there is a $q \in \mathbb{Q}$ such that $|x - q| < \frac{1}{n}$. 
The General Definition of a Function

(A function is its graph)

Let $X$ and $Y$ be sets. The **Cartesian product** of these sets is the set of all ordered pairs of elements taken from $X$ and from $Y$:

$$X \times Y = \{(x,y) : x \in X \text{ and } y \in Y\}$$

A **function between $X$ and $Y** is a nonempty subset of $X \times Y$ such that

$$\text{if } (x,y) \in f \text{ and } (x,y') \in f \text{ then } y = y'$$  \hspace{1cm} (!)

The **domain** of $f$ is the set

$$\text{dom}(f) = \{x \in X : \exists y \in Y \text{ s.t. } (x,y) \in f\}$$

The **range** of $f$ is the set

$$\text{ran}(f) = \{y \in Y : \exists x \in X \text{ s.t. } (x,y) \in f\}$$

The set $Y$ is called the **codomain** of $f$.

When $\text{dom}(f) = X$, we say $f$ is a **function from $X$ to $Y$** and write

$$f : X \to Y$$

Because of (!) each $x \in \text{dom}(f)$ determines a **unique** $y \in Y$ such that $(x,y) \in f$. This unique $y$ is denoted by $f(x)$ and we are back to the more
familair notation for a function $f$: $y = f(x)$. 
Basic Properties a Function May Have

A function \( f : X \rightarrow Y \) is **onto** (or **surjective**) if \( \text{ran}(f) = Y \).

A function \( f : X \rightarrow Y \) is **one-to-one** (or **injective**) if
\[
x \neq x' \Rightarrow f(x) \neq f(x')
\]
Equivalently \( f : X \rightarrow Y \) is one-to-one if
\[
f(x) = f(x') \Rightarrow x = x'
\]
A function \( f : X \rightarrow Y \) that is both injective and surjective is **bijective**.

If a function \( f : X \rightarrow Y \) is **bijective** we can define a new function \( g : Y \rightarrow X \) by
\[
g = \{(y,x) \in Y \times X : (x,y) \in f\}
\]
By definition
\[
\text{dom}(g) = \text{ran}(f) \text{ and } \text{ran}(g) = \text{dom}(f)
\]
and
\[
x = g(y) \iff y = f(x)
\]
Equivalently,
\[
f(g(y)) = y \text{ and } g(f(x)) = x
\]
for all \( x \in \text{dom}(f) \) and \( y \in \text{dom}(g) \). Because of these relations \( g \) is called the **inverse (function) of** \( f \) and
usually is denoted by $f^{-1}$. 
Real-valued Functions of a Real Variable

A function \( f : X \rightarrow Y \) with \( X \subseteq \mathbb{R} \) and \( Y \subseteq \mathbb{R} \) is a real-valued function of a real variable. For such functions we define:

\( f : X \rightarrow Y \) is **increasing** if
\[
x, x' \in \text{dom}(f) \text{ and } x < x' \Rightarrow f(x) \leq f(x')
\]

\( f : X \rightarrow Y \) is **decreasing** if
\[
x, x' \in \text{dom}(f) \text{ and } x < x' \Rightarrow f(x) \geq f(x')
\]

\( f : X \rightarrow Y \) is **strictly increasing** if
\[
x, x' \in \text{dom}(f) \text{ and } x < x' \Rightarrow f(x) < f(x')
\]

\( f : X \rightarrow Y \) is **strictly decreasing** if
\[
x, x' \in \text{dom}(f) \text{ and } x < x' \Rightarrow f(x) > f(x')
\]

\( f \) is **monotone** if it is either increasing or decreasing and is **strictly monotone** if it is either strictly increasing or strictly decreasing.