RIEMANN INTEGRATION

Throughout our discussion of Riemann integration
\[ B = B[a, b] = B([a, b], \mathbb{R}) \]
is the set of all bounded real-valued functions on close, bounded, nondegenerate interval \([a, b]\).

1. **DEF.** A finite set of points \( P = \{x_0, x_1, \ldots, x_n\} \) is a **partition of** \([a, b]\) if
   \[ a = x_0 < x_1 < \cdots < x_n = b. \]
We sometimes write
\[ P = \{a = x_0 < x_1 < \cdots < x_n = b\}. \]
The set of all partitions of \([a, b]\) is denoted by \( \mathcal{P} \).

2. **DEF.** A partition \( Q \) of \([a, b]\) is a **refinement** of a partition \( P \) of \([a, b]\) if \( P \subset Q \). Then \( Q \) is said to be **finer** than \( P \).

3. **DEF.** Let \( f \in B[a, b] \) and \( P = \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\). Then the \( i \)-th **subinterval of the partition** is
   \[ I_i = [x_{i-1}, x_i], \quad |I_i| = x_i - x_{i-1} \]
and
   \[ m_i = m_i(f) = \inf_{x \in I_i} f(x) \]
   \[ M_i = M_i(f) = \sup_{x \in I_i} f(x) \]

Then
\[ L_P(f) = \sum_{i=1}^{n} m_i |I_i| \]
is the **lower sum of** \( f \) **with respect to** the partition \( P \) and
\[ U_P(f) = \sum_{i=1}^{n} M_i |I_i| \]
is the **upper sum of** \( f \) **with respect to** the partition \( P \).

Facts:
1. For any partition $P$, $L_P(f) \leq U_P(f)$.

2. If $Q$ refines $P$, then
   
   $$L_P(f) \leq L_Q(f) \quad \text{and} \quad U_Q(f) \leq U_P(f)$$

3. If $P$ and $Q$ are any two partitions of $[a,b]$, then
   
   $$L_P(f) \leq U_Q(f).$$

4. DEF. Let $f \in B[a,b]$. The lower integral of $f$ with respect to the partition $P$ is

   $$(L) \int_a^b f(x) \, dx = L(f) = \sup_{\mathcal{P}} L_\mathcal{P}(f)$$

   and the upper integral of $f$ with respect to the partition $P$ is

   $$(U) \int_a^b f(x) \, dx = U(f) = \inf_{\mathcal{P}} U_\mathcal{P}(f)$$

5. Fact:

   $$\int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx$$

   $$L(f) \leq U(f)$$

6. DEF. Let $f \in B[a,b]$. Then $f$ is Riemann integrable (R-integrable) on (over) $[a,b]$ if $L(f) = U(f)$ in which case the Riemann integral of $f$ on (over) $[a,b]$ is

   $$\int_a^b f(x) \, dx = L(f) = U(f).$$

   Notation: $\mathcal{R}[a,b]$ is the set of all R-integrable functions on $[a,b]$.

7. Facts:
1. Let $c \in \mathbb{R}$. The function $f : [a, b] \to \mathbb{R}$ defined by $f(x) = c$ is integrable over $[a, b]$ and
   \[ \int_a^b c \, dx = c(b - a) \]

2. (Dirichlet Function) The function $f$ defined by
   \[ f(x) = \begin{cases} 
   1 & \text{for } x \in [a, b] \cap \mathbb{Q} \\
   0 & \text{for } x \in [a, b] \cap \mathbb{I}
   \end{cases} \]
   is not Riemann integrable over $[a, b]$.

Existence of the Integral

8. Th. Let $f \in \mathcal{B}[a, b]$. Then $f$ is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P$ such that
   \[ U_P(f) - L_P(f) < \varepsilon. \]

9. Th. If $f : [a, b] \to \mathbb{R}$ is monotone, then $f$ is integrable on $[a, b]$.

10. Th. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f$ is integrable on $[a, b]$.

11. (Interval Additivity) Let $[a', b'] \subset [a, b]$ and $c \in [a, b]$. Then
    \[ f \in \mathcal{R}[a, b] \implies f \in \mathcal{R}[a', b'], \]
    \[ f \in \mathcal{R}[a, c] \text{ and } f \in \mathcal{R}[c, b] \implies f \in \mathcal{R}[a, b] \text{ and } \]
    \[ \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \]
    Note: A direct consequence of 9, 10, and 11 is: If $f$ is defined on an interval $[a, b]$ that can be decomposed into adjacent subintervals on each of which $f$ is either monotone or continuous, then $f$ is integrable over the full interval $[a, b]$.

12. Th. (Order Properties) If $f \in \mathcal{R}[a, b]$ and $f \geq 0$ on $[a, b]$, then
    \[ \int_a^b f(x) \, dx \geq 0. \]
Cor. If \( f, g \in \mathcal{R}[a, b] \) and \( f \geq g \) on \([a, b] \), then
\[
\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx.
\]

Cor. If \( f \in \mathcal{R}[a, b] \) and \( m \leq f \leq M \) on \([a, b]\), then
\[
m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a).
\]

Combinations of Integrable Functions

13. (Linearity) If \( f, g \in \mathcal{R}[a, b] \) and \( \alpha \in \mathbb{R} \), then
\[
f + g \in \mathcal{R}[a, b] \quad \text{and} \quad \alpha f \in \mathcal{R}[a, b].
\]

14. If \( f, g \in \mathcal{R}[a, b] \) then
\[
\begin{align*}
f^2 & \in \mathcal{R}[a, b] \\
gf & \in \mathcal{R}[a, b]
\end{align*}
\]

15. (Triangle Inequality for Integrals) If \( f \in \mathcal{R}[a, b] \) then \( |f| \in \mathcal{R}[a, b] \) and
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx
\]

16. (Intermediate Value Theorem for Integrals) If \( f \in \mathcal{C}[a, b] \) and \( g \in \mathcal{R}[a, b] \) is positive on \([a, b]\), then there is a point \( c \in [a, b] \) such that
\[
\int_a^b f(x) g(x) \, dx = f(c) \int_a^b g(x) \, dx.
\]

Cor: If \( f \in \mathcal{C}[a, b] \), then there is a point \( c \in [a, b] \) such that
\[
\int_a^b f(x) \, dx = f(c)(b-a).
\]
17. **DEF:** In writing \( f \in \mathcal{R} [a, b] \) it was assumed that \( a < b \). Then by definition

\[
\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx
\]

and

\[
\int_a^a f(x) \, dx = 0.
\]

**Remarks:**

1. These equalities become theorems if we adjust our previous definitions so that a partition \( P = \{a = x_0, x_1, \ldots, x_n = b\} \) where the intermediate points in the partition increase if \( a < b \) and decrease if \( b < a \). When \( a = b \) there is only one partition which is \( \{a = x_0, b = x_n\} \).

2. With these extensions the order properties only hold when \( a < b \).

3. The other results hold for any \( a \) and \( b \) and interval additivity is true for any order of the points \( a, b, \) and \( c \).

4. It is also useful to note that regardless of the order of \( a \) and \( b \),

\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx
\]

**The Fundamental Theorem (FTC) of Calculus**

Throughout this topic \( f : I \to \mathbb{R} \) is a function defined on an interval of any type and \( f \in \mathcal{R} [a, b] \) for every closed, bounded subinterval \([a, b] \subset I\). Also, \( c \in I \) is fixed and \( F : I \to \mathbb{R} \) is the function defined by

\[
F(x) = \int_c^x f(t) \, dt.
\]

18. If \( f : I \to \mathbb{R} \) is as described, then \( F(x) = \int_c^x f(t) \, dt \) is continuous on \( I \).

19. *(FTC I)* If \( f : I \to \mathbb{R} \) is as described, then \( F(x) = \int_c^x f(t) \, dt \) is differentiable at each point \( x \in I \) at which \( f \) is continuous and

\[
F'(x) = f(x).
\]
20. *(FTC II)* If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous and \( F \) is any antiderivative of \( f \) on \( I \), then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

21. *(FTC III)* If \( f' \in \mathcal{R}[a, b] \), then
\[
\int_a^b f'(x) \, dx = f(b) - f(a).
\]

**Convergence and the Integral**

22. **Th. Th.** If \( f_n \in \mathcal{R}[a, b] \) and \( f_n \) converges uniformly to \( f \) on \([a, b]\), then \( f \in \mathcal{R}[a, b] \) and
\[
\int_a^b f_n(x) \, dx \rightarrow \int_a^b f(x) \, dx
\]
as \( n \rightarrow \infty \), equivalently,
\[
\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) \, dx.
\]

**The Riemann Sum Connection**

23. **DEF.** Let \( P = \{a = x_0, x_1, ..., x_n = b\} \) be a partition of \([a, b]\) and \( c_i \in I_i \). Then
\[
\sum_{i=1}^n f(c_i) |I_i|
\]
is a **Riemann sum of \( f \) over (on) \([a, b]\)**.

23. **Th.** Let \( f \in \mathcal{B}[a, b] \). Then \( f \) is R-integrable over \([a, b]\) if and only if there is a real number \( A \) such that for every \( \varepsilon > 0 \) there is a partition \( P \) of \([a, b]\) such that
\[
\left| A - \sum_{i=1}^n f(c_i) |I_i| \right| < \varepsilon
\]
for all choices of \( c_i \in I_i \).