Lesson 6

The Collective Model and the Fermi Gas Model
Evidence for Nuclear Collective Behavior

- Existence of permanently deformed nuclei, giving rise to “collective excitations”, such as rotation and vibration
- Systematics of low lying 2+ states in many nuclei
- Large transition probabilities for 2+ $\rightarrow$ 0+ transitions in deformed nuclei
- Existence of giant multipole excitations in nuclei
Deformed Nuclei

- Which nuclei?  \( A = 150-190, \ A > 220 \)
- What shapes?
How do we describe these shapes?

\[ R(\theta, \phi) = R_{\text{avg}} \left[ 1 + \beta Y_{20}(\theta, \phi) \right] \]

\[ \beta = \frac{4}{3} \sqrt{\frac{\pi}{5}} \frac{b - a}{R_{\text{avg}}} \]

\[ R_{\text{avg}} = \frac{1}{2} \left( a^2 + b^2 \right) \]
Rotational Excitations

- Basic picture is that of a rigid rotor

\[ E_{\text{rot}} = \frac{J(J + 1)\hbar^2}{2\mathcal{S}} \]

where

\[ \mathcal{S}_{\text{sphere}} = \frac{2}{5} mR^2 \]

\[ \mathcal{S}_{\text{ellipsoid}} = \frac{2}{5} mR^2 \frac{1}{(1 + 0.31\beta)} \]

- | \( \text{J} \) | \( \text{Energy} \) |
  | 0+   | 4.14  |
  | 1+   | 3.50  |
  | 2+   | 2.88  |
  | 3+   | 2.30  |
  | 4+   | 1.75  |
  | 5+   | 1.23  |
  | 6+   | 0.76  |
  | 7+   | 0.34  |
  | 8+   | 0.0   |
Are nuclei rigid rotors?

- No,
  \[ \mathcal{I}_{\text{irrot}} < \mathcal{I}_{\text{exp}} < \mathcal{I}_{\text{rigid}} \]
Backbending
Rotations in Odd $A$ nuclei

• The formula on the previous slide dealt with rotational excitations in $e^{-}e^{+}$ nuclei.

• What about odd $A$ nuclei?

• Here one has the complication of coupling the angular momentum of the rotational motion and the angular momentum of the odd nucleon.
Rotations in Odd A nuclei (cont.)

For $K \neq 1/2$

$$E(I) = \frac{\hbar^2}{2\alpha} [I(I+1) - K(K+1)]$$  \quad I=K, K+1, K+2, \ldots$$

For $K=1/2$

$$E_{K=1/2}(I) = \frac{\hbar^2}{2\alpha} [I(I+1) - \alpha(-1)^{I+1/2}(I+1/2)]$$  \quad I=1/2, 3/2, 5/2, \ldots$$
Nuclear Vibrations

- In analogy to molecules, if we have rotations, then we should also consider vibrational states
- How do we describe them?
- Most nuclei are spherical, so we base our description on vibrations of a sphere.
Shapes of nuclei

shaded areas are deformed nuclei

Types of vibrations of spherical nuclei

Figure 5.13 The first vibration modes of the nuclear surface, showing the form of the nucleus for each mode (solid line) in comparison to the original spherical nucleus. The dotted line represents the original “fixed volume” sphere.
Formal description of vibrations

\[ R(\theta,t) = R_0 \left[ 1 + \sum_{\lambda=0}^{\infty} \alpha_{\lambda,0}(t) Y_{\lambda,0}(\cos \theta) \right] \]

Suppose \( \lambda=0 \)

\[ R(\theta,t) = R_0 \left[ 1 + \alpha_{0,0}(t) / \sqrt{4\pi} \right] \]

Suppose \( \lambda=1 \)

\[ R(\theta,t) = R_0 \left[ 1 + \sqrt{\frac{3}{4\pi}} \alpha_{1,0}(t) \cos \theta \right] \]

Suppose \( \lambda=2 \)

\[ R(\theta,t) = R_0 \left[ 1 + \alpha_{2,0}(t) \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \right] \]
Nuclear Vibrations

Figure 5.13 The first vibration modes of the nuclear surface, showing the form of the nucleus for each mode (solid line) in comparison to the original spherical nucleus. The dotted line represents the original “fixed volume” sphere.
Levels resulting from these motions

Hexadecapole \((2^4)\) \(4^+\)

Octupole \((2^3)\) \(3^-\)

- 3-phonon
  - 0, 2, 3, 4, 6
- 2-phonon
  - 0, 2
  - 2
- 1-phonon
  - 2

Pure vibrational (a)

Vibrational states in real nuclei (b)

Higher-order vibrations (c)
You can build rotational levels on these vibrational levels

\[ E = \frac{\hbar^2}{2S} \left[ J(J + 1) - K^2 \right] \]
Net Result
Nilsson model

• Build a shell model on a deformed h.o. potential instead of a spherical potential
Successes of Nilsson model

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>Z</th>
<th>N</th>
<th>Exp.</th>
<th>Shell</th>
<th>Nilsson</th>
</tr>
</thead>
<tbody>
<tr>
<td>19F</td>
<td>9</td>
<td>10</td>
<td>1/2+</td>
<td>5/2+</td>
<td>1/2+</td>
</tr>
<tr>
<td>21Ne</td>
<td>10</td>
<td>11</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>21Na</td>
<td>11</td>
<td>10</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>23Na</td>
<td>11</td>
<td>12</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>23Mg</td>
<td>12</td>
<td>11</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
</tbody>
</table>
Real World Nilsson Model
Fermi Gas Model

• Treat the nucleus as a gas of non-interacting fermions
• Suitable for predicting the properties of highly excited nuclei
Details

Consider a box with dimensions $L_x, L_y, L_z$ in which particle states are characterized by their quantum numbers, $n_x, n_y, n_z$. Change our descriptive coordinates to $p_x, p_y, p_z$, or their wavenumbers $k_x, k_y, k_z$ where $k_i = n_i \pi / L_i = p_i / \hbar$.

The highest occupied level $\equiv$ Fermi level.

$$
\begin{align*}
  k_f^2 &= k_x^2 + k_y^2 + k_z^2 \\
  \frac{k_f^2 L^2}{\pi^2} &= n_x^2 + n_y^2 + n_z^2 \\
  E(n_x, n_y, n_z) &= \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \\
  \text{Imposing quantization conditions}
\end{align*}
$$

$$
N_{\text{states}} = \left( \frac{1}{8} \right) \frac{4\pi}{3} \left( \frac{kL}{\pi} \right)^3
$$
Making the box a nucleus

\[
k_f = \frac{\pi}{L} \left( \frac{2N_{\text{states}}}{3\pi} \right)^{1/3} = \frac{\pi}{L} \left( \frac{2Z}{3\pi} \right)^{1/3} = \frac{\pi}{r_0} \left( \frac{2Z}{3\pi A} \right)^{1/3}
\]

Then

\[
E_f = \frac{k_f^2 \hbar^2}{2m} \approx 32 \text{ MeV}
\]

\[
E_f^{\text{protons}} = 53 \left( \frac{Z}{A} \right)^{1/3} \text{ MeV}
\]

\[
E_f^{\text{neutrons}} = 53 \left( \frac{A - Z}{A} \right)^{1/3} \text{ MeV}
\]
Thermodynamics of a Fermi gas

- Start with the Boltzmann equation
  \[ S = k_B \ln \Omega \]

- Applying it to nuclei

\[
S = k \ln \Omega = k \{ \ln[\rho(E^*)] - \ln[\rho(0)] \}
\]

\[
S = \int_0^T \frac{dE^*}{T}
\]

\[
E^* = a(kT)^2
\]

\[
dE^* = 2ak^2TdT
\]

\[
S = 2ak^2 \int_0^T dT = 2ak^2T = 2k(aE^*)^{1/2}
\]

\[
\frac{\rho(E^*)}{\rho(0)} = \exp \left( \frac{S}{k} \right) = \exp \left[ 2(aE^*)^{1/2} \right]
\]
Relation between Temperature and Level Density

• Lang and LeCouteur

\[ \rho(E) = \frac{1}{12} \left( \frac{\pi^2}{a} \right)^{1/4} \frac{1}{(E + T)^{5/4}} \exp[2(aE)^{1/2}] \]

\[ E = aT^2 - T \]

\[ a = A/10 \]

• Ericson

\[ \rho(E) = \frac{1}{12} \left( \frac{\pi^2}{a} \right)^{1/4} \frac{1}{(E)^{5/4}} \exp[2(aE)^{1/2}] \]

\[ E = aT^2 \]

\[ a = A/10 \]
What is the utility of all this?