Title: Measuring the Effects of Non Economic Constraints on Welfare

Authors: Ryan Siegel[1] and Robin Cross[2]

Abstract: Consumer purchase decisions are constrained by factors beyond the classic budget constraint, such as time, weather, rationing, or other institutional factors. We compare time and income constraints on store-level demand using a nonparametric capacity utilization framework to estimate welfare loss. Using data for fifteen nursery products held at hundreds of stores in the western United States we find losses from time constraints to be larger than from income. Robustness checks accounting for weather, product types, and aggregation confirm results.

Keywords: time constraint, capacity utilization, nursery industry, welfare

Introduction

How do economic vs. non-economic constraints such as time, weather, or institutional factors, affect welfare? By better understanding the effects of non-economic constraints on sales, businesses may increase their efficiency and productivity through improved production decisions and inventory planning. In this paper, we separate and compare the impacts of time and income constraints on sales using live goods as an example. We show that for some goods, time may

[1] Agricultural and Resource Economics Department, 213 Ballard Extension Hall, Oregon State University, Corvallis, Oregon 97370, United States of America. siegelry@onid.orst.edu, 541 602-1548 (correspondence author)

[2] Agricultural and Resource Economics Department, 213 Ballard Extension Hall, Oregon State University, Corvallis, Oregon 97370, United States of America. robin.cross@oregonstate.edu, 541 737-1397

have a greater impact than income; especially during fall and winter when other activities compete for gardening time.

BLANK popularized the use of the income constraint with utility theory to achieve closed-form demand solutions. In response to wartime rationing, economists derived demand under two or more constraints and explored elasticity results (for reviews see Tobin, 1952; Jackson, 1991; Hanemann, 2004). Besides rationing, other demand constraints have interested economists. Becker (1965) elaborated a theory of household production under time and income constraints but uses a time-price to reduce the problem to a single 'full-income' constraint. DeSerpa (1971) and Steedman (2001) extend the classical theory of household demand to have a time constraint but, like Becker, neglect discussing firms.

Samuelson shed light on constrained demand and equilibrium. He proposed a model of demand under multiple, irreducible constraints (1947, pp 163-171) and showed that, in equilibrium, a constraint's (α) effect on equilibrium quantity, x_i, is

(1) $$\partial x_i / \partial \alpha = |A_i| / |A|$$

where $|A|$ is the determinant of the excess demand Jacobian and $|A_i|$ is the determinant of the A matrix with its i^{th} column replaced with partial derivatives of inverse-demands with respect to the constraint (ibid. p 259) .

Closed form solutions for (1) are unlikely to exist. Färe *et al*. (1989), however, developed a nonparametric method to estimate output loss from a constraint, without the use of shadow prices, based on Johansen's (1968) concept of production capacity—i.e. the maximum producible amount given an unrestricted variable input. By sequentially relaxing variable input constraints, frontier losses attributable to each input can be measured using data envelopment analysis (DEA).

In the next section, we propose using the Färe *et al*. (1989) capacity utilization model as a non-parametric estimate of the effect of time and income in equilibrium. Next, we introduce datasets on time spent gardening, personal incomes, and sales/inventories of plant material in hundreds of garden centers across the United States. In section 3 we estimate the relative impacts of time and income and test for significant differences using a subsample bootstrap by region, product type, and quarter. The paper concludes with summary findings and recommendations for further empirical and theoretical research.

Model

Denote the output vector $y \in \mathcal{R}_{+}^{M}$, input vector $x \in \mathcal{R}_{+}^{N}$ and technology $T = \{(x, y): x \ can \ produce \ y\}$. Technology is assumed to be convex and freely disposable. Following Färe & Grosskopf (2000), the directional distance function is the distance from an observation to the production frontier:

(2) $$\vec{D}_T(x, y; -g_x, g_y) = \sup\{\beta: (x - \beta g_x, y + \beta g_y) \in T\}$$

where g_x and g_y are direction vectors.

Welfare loss is the difference between the constrained and unconstrained frontier. We assume non increasing returns to scale and avoid zero distance measures by measuring in the direction of the k^{th} observation's output vector[3]. We treat time and income as nondiscretionary by measuring in the output direction only (Thanassoulis, Portela & Despic, 2008 pg 345). Finally, denote the set of input vectors *without* time or income as x_t and as x_i, respectively. Distance functions for each case are denoted $\widehat{D}_T\left(x_j, y; -g_{x_j}, g_y\right)$ $j = t, i$. The measure for welfare loss from time or income thus becomes:

(3) $$F_j = \widehat{D}_T\left(x_j, y; 0, y\right) - \vec{D}_T(x, y; 0, y) \;\; j = t, i$$

Since the original producible output set is a subset of the one with no restriction on variable inputs, we have $F_j \geq 0$ $j = t, i$ (Färe, Chenggang, & Seavert, 2010). The difference of welfare loss from time versus income is denoted $F_{ti} = F_t - F_i$ and may thus be positive or negative.

We estimate the distance function (1) using the linear programming problem used by Färe & Grosskopf (2000) under the required maximum feasibility constraints (Färe, *et al*., 2008; see Appendix I).

Confidence intervals for the mean welfare measures are constructed using a subsample bootstrapping procedure (Campbell, *et al*., 1997). The bootstrapping procedure iterates 25 times with subsamples sized 40% of the original for computing speed. We then test whether time has a

[3] If measuring in the direction of all outputs with, say, vector $g_y = \vec{1}$, and if the frontier for at least one of the outputs is zero, then the scaling factor will equal zero.

greater effect on sales than income. The null and alternate hypotheses are constructed so that rejecting the null hypothesis implies that time has a greater effect on sales loss than income, or:

(4) $$H_0: \overline{F_{t\iota}} \leq 0$$

Data

We define the decision making unit (DMU) to be a single community which converts inputs (time, income, population density, and product inventories) into a series of outputs (product sales) in a given week. Product inventory and sales volume data come from 2009 scanner data (Green Market Systems Inc., 2010) for 15 products in 193 stores in the western United States and are grouped into five categories: Annuals, Azaleas, Cypress trees, Succulents, and Vegetables (see Appendix II).

Use of scanner data has its limitations. Returns introduce downward bias in sales and upward bias in inventories. However, not all inventory is recorded when received, thereby introducing a downward bias. Records with negative sales volumes (i.e. returns exceed purchases) are precluded from the analysis. If we assume that frontier and interior data points are similarly negatively biased, then our FST and FSI measures are upwards-biased since they are now a fraction of less input.

The Bureau of Labor Statistics (BLS) provides data for the store's community income and time inputs. Wage data is collected under the Quarterly Census of Employment and Wages Program at the quarterly, county level (BLS, 2011a) and serves as a proxy for community income per week. The BLS also provides data from the American Time Use Survey (ATUS) which tabulates thousands of time use diaries for individuals aged 15 and over participating in the Current Population Survey (BLS, 2011b). A state/month weighted average of time spent gardening is calculated using ATUS population weights, and serves as a proxy for a community's gardening time per day.

Time spent gardening fluctuates seasonally, with spring months busiest (see Appendix III). To control for climactic variation, we conduct our analysis within ten temperature/precipitation regions created using NOAA climate maps (NOAA, 2010). We split regions using three precipitation ranges (mean/yearly total precipitation in inches) and four temperature ranges (mean daily average temperature). Figure 1, below, shows the stores in the final dataset and their placement in the western United States. Appendix IV shows the regions across the continental US.

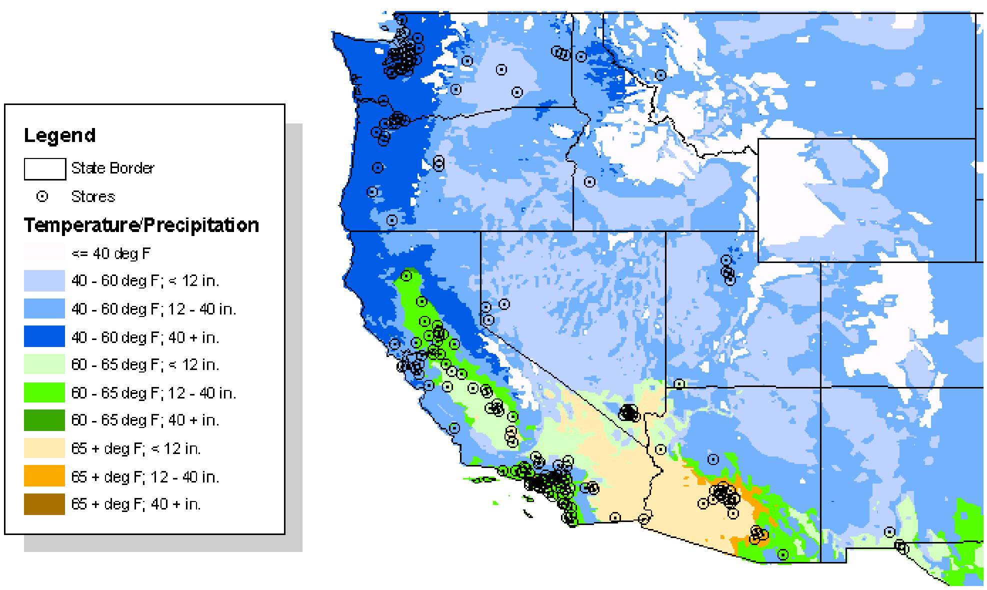


Figure 1 - Stores and Temperature/Precipitation Regions

Nearly every store is located in a separate zip code tabulation area (ZCTA); each with a distinct population density. Data is provided by the US Census bureau (2010). In our sample, population density ranges from 7 to 5250 people per square kilometer and averages about 926 (see Appendix V).

The inventory, sales, population density, time, and income data for 2009 combine to produce 7 regional data sets[4]; each with 1109 observations (community/weeks) on average. Appendix VI provides summary statistics on inputs and outputs used by region. Within any region, nearly every variable except income has sizeable standard deviations indicating a good degree of heterogeneity in the data. The table in Appendix VI indicates that stores may not carry inventories for any of the products in each category. Zero inventories/sales is especially pronounced for cypress trees which are held in a given store/week only 25% of the time in regions 4, 5, 7 and 8.

Results

Initial results are presented in Table 1. The bootstrapped $\overline{F_t}$ measure averages 10.7%, i.e. the average store loses 10.7% in sales due to people's limited time[5]. On the other hand, welfare loss from the income constraint averages only 0.36%. Both time and income had the greatest impact

---

[4] Temperature/precipitation regions 0, 6 and 9 have no stores in our sample.

[5] The careful reader will note that we are measuring differences of the frontiers, not actual sales, per se. This is correct. However, firms are not far from the frontier since their sales average about 94% of their respective frontier sales.

in regions 2 and 3, among the cooler and wetter climate regions. Time also appears to have greatest impact in the wealthier areas (Region 2) and less pronounced in temperate areas (regions 4 and 5). Altogether, this suggests that time matters much more for nursery-product decisions than does income. Indeed, the mean difference measure, $\overline{F_{t\iota}}$, is significantly positive at the 97.5% confidence level for six out of the seven regions averaging 10.3%. Thus, time inhibits sales, on average, 10.3% more on average than does income.

Table 1 – Percent of Sales Lost due to Constraints*

	Region 1	Region 2	Region 3	Region 4	Region 5	Region 7	Region 8
Subsample Size	*166*	*591*	*334*	*459*	*932*	*558*	*47*
Time ($\overline{F}_t$)	14.9 (49.1, 2)	23.3 (58.5, 8)	24.9 (41.5, 9.7)	4.4 (7.6, 2)	6.3 (9.4, 3.6)	16.9 (23.7, 11.7)	0.8 (4, 0)
Income ($\overline{F}_\iota$)	0.1 (0.3, 0)	0.3 (1.1, 0.1)	1 (6.4, 0.1)	0.1 (0.3, 0)	0.4 (0.6, 0.1)	0.3 (0.5, 0.1)	0 (0.2, 0)

* *Table presents bootstrapped mean and 95% confidence intervals; units in percent of current firm production*

Table 2 summarizes bootstrap results for the bootstrapped difference measure by region *and* product category[6]. Results suggest that time has a greater effect than income across product types and across most regions. The insignificant results in regions 1 and 8 may be due to smaller sample sizes.

Table 2 – Difference between Time and Income Effect on Sales by Region and Product Type*

	Region 1	Region 2	Region 3	Region 4	Region 5	Region 7	Region 8
Subsample	*166*	*591*	*334*	*459*	*932*	*558*	*47*
All	**14.8 (49, 2)**	**23 (58, 7.9)**	**23.9 (39.9, 9.4)**	**4.3 (7.5, 2)**	**5.9 (8.9, 3.1)**	**16.6 (23.5, 11.5)**	0.7 (3.9, -0.1)

[6] To control for product type, the distance function was measured in the relevant direction. In order to avoid unbounded results, store/weeks with no sales in the product category are excluded from the analysis.

Vegetables	0.4 (2.7, -0.2)	**16.3** **(50.1, 3.1)**	10.3 (40, -3.6)	**8.3** **(16.5, 2.6)**	**13.4** **(20.4, 5.7)**	**37.4** **(55.4, 18.5)**	2.5 (13.3, -0.5)
Succulents	6.3 (23.9, -0.1)	**14.2** **(30.1, 7.6)**	**16.8** **(35.8, 5.8)**	**11** **(17.7, 4.2)**	**16.8** **(26.6, 10.6)**	**50.4** **(78.2, 32.3)**	2.6 (16.1, -0.3)
Azaleas	0.4 (3.1, 0)	**77.1** **(138.3, 32.6)**	**67.3** **(167, 27)**	**13.8** **(35, 1.6)**	**15.4** **(39.1, 7.9)**	**20.2** **(46.1, 1)**	2.1 (13, -0.1)
Annuals	**9.1** **(35.4, 0)**	**33.7** **(104.3, 12.7)**	**17.7** **(73, 1.8)**	**7.6** **(12.6, 4)**	**18.8** **(33.3, 11.2)**	**76.6** **(185.3, 42.4)**	2.5 (9.5, -0.5)
Cypress	**27** **(203.4, 1.4)**	**62** **(115.6, 24.8)**	**42.3** **(78.4, 14)**	1.9 (6.7, -0.1)	**6.8** **(12.4, 0.5)**	0.6 (9.7, 0)	0 (0, 0)

* *Table presents bootstrapped mean and 95% confidence intervals; units in percent of current firm production; boldface indicates significantly positive*

Examining the difference measure by quarter reveals pronounced heterogeneity. Table 3 shows that time is a more significant deterrent to sales particularly in the first and last three months of the year—regardless of climate.

Table 3 - Difference between Time and Income Effect on Sales by Region and Quarter *

	Region 1	Region 2	Region 3	Region 4	Region 5	Region 7	Region 8
All	**14.8** **(49, 2)**	**23** **(58, 7.9)**	**23.9** **(39.9, 9.4)**	**4.3** **(7.5, 2)**	**5.9** **(8.9, 3.1)**	**16.6** **(23.5, 11.5)**	0.7 (3.9, -0.1)
Qr 1	**7.8** **(24.8, 0)**	**24.3** **(57.2, 4.4)**	**21.2** **(51.7, 4.5)**	**3.3** **(7.4, 0.8)**	**7.5** **(12, 2.7)**	**2.2** **(4.3, 0.4)**	0.9 (3.9, 0)
Qr 2	0 (0.1, 0)	0.1 (0.5, -0.1)	0.3 (1.7, -0.4)	0 (0.1, -0.1)	0 (0.1, -0.2)	**1.1** **(3.5, 0)**	n.d.**
Qr 3	0.3 (1.9, -0.1)	**0.7** **(1.7, 0)**	0.3 (1.4, 0)	0.3 (0.8, -0.1)	0.1 (0.3, -0.1)	2.8 (8.6, -0.1)	n.d.
Qr 4	1.3 (6.3, 0)	**23.7** **(61.1, 4.3)**	**23.8** **(67.6, 2.6)**	**3.6** **(8.3, 1.2)**	**6.3** **(8.9, 3.7)**	**33.2** **(56.3, 15)**	1.1 (7.4, 0)

* *Table presents bootstrapped mean and 95% confidence intervals; units in percent of current firm production; boldface indicates significantly positive* ** *n.d. indicates no data;*

This difference by quarter or season makes intuitive sense. Spring and summer months are among the busiest in terms of time spent gardening. The results above might suggest, then, that during the spring and summer months, people prioritize gardening such that more time is unnecessary. In the winter and fall, however, other activities are prioritized such that time spent gardening diminishes. The availability of extra time to garden during these times, therefore, becomes more valuable to customers, which is reflected in the significantly positive results for quarters 1 and 4. There is, in effect, an interaction between a seasonality-constraint and a time constraint.

To test sensitivity to aggregation levels, we re-aggregate data to the store-quarter level. Time is aggregated to the state/quarterly levels instead of state/month. Population density and income data remains unchanged. Aggregation reduces the size of the data set to Z community/quarter observations. Given the high number of inputs and outputs, we examine the three central measures for Z regions.

Aggregation at the quarterly level does not impact results with the exception of…

All together, results suggest that people's time and income both inhibit live good sales. The gardening time constraint reduces sales by approximately 10.7% whereas the income constraint does so by only 0.4%. The difference is found to be significantly positive for five out of the seven regions and particularly for the first and last quarters of the year. These results confirm a recent study, albeit in a very different setting, which found that time is more constraining than money in determining people's achievement of a minimum nutritious diet (Davis & You, 2011).

Conclusions:

This paper extends Färe, *et al.*'s capacity utilization measure to a short-run equilibrium framework to measure welfare loss due to demand constraints. In this way we estimate welfare loss in a given market using a single measure without the use of a constraint shadow price.

We find that time is often a greater constraint than the classical budget constraint. This suggests that businesses should consider people's non-monetary constraints. Specifically, nursery businesses may need to pay more attention to addressing people's time limitations during the winter and fall months when their gardening-time is more valuable.

Recommendations for future research vary from the empirical to the theoretical. Empirically, it would be useful to study the same question but in different market settings. It would also be an improvement to use a dataset with greater income heterogeneity. Estimating a similar model to the one in this paper econometrically would provide further support to the argument. Finally, a similar analysis could be conducted using constraints other than time such as weather.

Theoretical foundations of consumers constrained by multiple constraints should also be further explored. Theoretical work is needed to support the notion of welfare loss from non-economic constraints. Also, ways of incorporating multiple constraints in demand specification should be investigated especially as it pertains to issues of aggregation and econometric estimation. Results suggest that constraints come in and out of being binding which may present issues in econometric demand estimation.

References

Becker, G.S. (1965). A Theory of the Allocation of Time. The Econ. J., Vol. 75, No. 299 September, pp. 493-517

Bureau of Labor Statistics. (2011a). Quarterly Census of Employment and Wages, 2009, Quarters 1 thru 4, Single File [Data file]. Available from http://www.bls.gov/cew/

Bureau of Labor Statistics. (2011b). American Time Use Survey Respondent, Roster, Activity files [Data files]. Available from http://www.bls.gov/tus/

Campbell, John Y., Lo, Andrew, W., & MacKinlay, A Craig. (1997). The Econometrics of Financial Markets. Princeton, NJ: Princeton University Press.

Davis, G.C & W. You. (2011). Not Enough Money or Not Enough Time to Satisfy the Thrifty Food Plan? A Cost Difference Approach for Estimating a Money-Time Threshold. Food Policy 36, 101-107.

Färe R., Grosskopf, S., & E.C. Kokkelenberg. (1989) Measuring Plant Capacity, Utilization and Technical Change: A Nonparametric Approach. International Economic Review, Vol. 30, No. 3 August, pp. 655-666

Färe, R., Grosskopf, S. (2000). Theory and Application of Directional Distance Functions. Journal of Productivity Analysis, (13) pp. 93–103.

Färe, R., Grosskopf, S., & Margaritis D. (2008). Efficiency and Productivity: Malmquist and More. In H.O., Fried, C.A.K., Lovell & S.S. Schmidt (Eds.), *The Measurement of Productive Efficiency and Productivity Growth* (522-621). New York, NY: Oxford University Press, Inc.

Green Market Systems, Inc. (2010). 2009 Scanner Data [Data File].

Johansen, L. (1968). Production Functions and the Concept of Capacity. Recherches récentes sur la fonction de production, collection “Economie Mathématique et Econometrie,” 2 pp 46-72.

National Oceanic and Atmospheric Administration. (2011). Climate Maps of the United States [Data File]. Available from http://cdo.ncdc.noaa.gov/cgi-bin/climaps/climaps.pl?directive=welcome&subrnum=

Samuelson, P. (1947). Enlarged ed., 1983. Foundations of Economic Analysis, Harvard University Press

Thanassoulis, Emmanuel, Portela Maria C.S., & Ozren Despic. (2008). Data Envelopment Analysis: The Mathematical Programming Approach to Efficiency Analysis. In H.O., Fried, C.A.K., Lovell & S.S. Schmidt (Eds.), *The Measurement of Productive Efficiency and Productivity Growth* (251-420). New York, NY: Oxford University Press, Inc.

US Census Bureau. (2010). Census 2000 U.S. Gazetteer Files [Data file]. Available from http://www.census.gov/geo/www/gazetteer/places2k.html

Appendix I

The linear programming problem adapted from by Färe & Grosskopf (2000) is as follows:

(A.1) $\vec{D}_T(x, y; \vec{0}, y) = \max_{z,\beta} \beta \quad s.t. \quad \sum_{\forall k} z_k x_{k,n} - x_{k',n} \le 0, \forall n = 1, \dots, N$

$$-\sum_{\forall k} z_k y_{k,m} + y_{k',m} + \beta y_{k',m} \le 0, \forall m = 1, \dots, M$$

$$z_k \ge 0, \forall k = 1, \dots, K; \sum_{k=1}^{K} z_k \le 1;$$

The left-side distance function in (2) is found using (A.1) except that the first inequality holds for fixed inputs only. Furthermore, to ensure a maximum is achieved in (A.1) we must assume (Färe, *et al*., 2008)[7]:

(A.2) $\sum_{k=1}^{K} x_{k,n} > 0 \;\; \forall n = 1, \dots, N; \sum_{n=1}^{N} x_{k,n} > 0 \;\; \forall k = 1, \dots, K$

(A.3) $\sum_{k=1}^{K} y_{k,m} > 0 \;\; \forall m = 1, \dots, M; \sum_{m=1}^{M} y_{k,m} > 0 \;\; \forall k = 1, \dots, K$ and

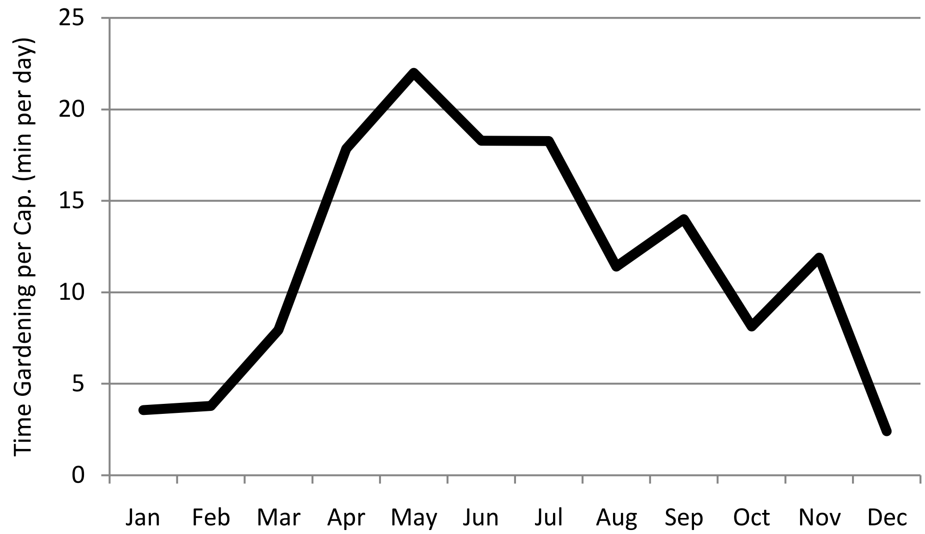
(A.4) $y_{k,n} \ge 0, x_{k,n} \ge 0, \;\; n = 1, \dots, N; \;\; m = 1, \dots, M; \;\; k = 1, \dots, K;$

Appendix II: Product Categories

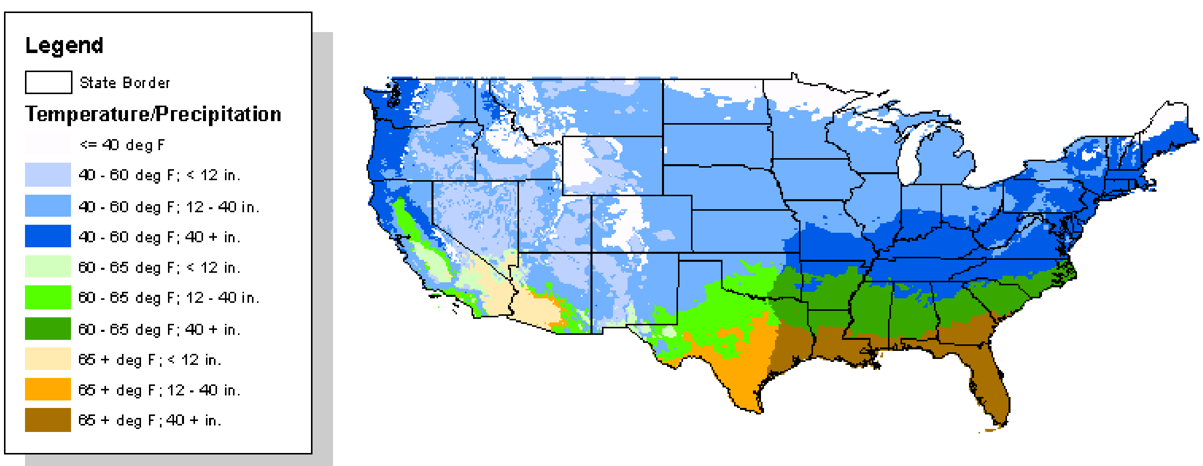
Product Category	Product Size/Name
Annuals	6pk Annuals
Annuals	1.25 qt Annual assorted
Azalea, flower	2.5 qt Azalea assorted
Azalea, flower	2.25 gal Azalea assorted
Cypress trees	2.25 gal Cypress leyland
Cypress trees	3.25 gal Arborvitae emerald green
Cypress trees	2.5 qt Arborvitae emerald green
Succulent, drought	4 oz Succulent assorted
Succulent, drought	13 oz Succulent assorted
Vegetables	2.5 qt Vegetables
Vegetables	1.0 gal Vegetable/herb planter
Vegetables	1.25 qt Vegetable/herbs
Vegetables	2.5 qt Rosemary
Vegetables	1.0 pt Strawberry
Vegetables	1.0 pt Herb assorted

[7] For (5) to hold in the fixed input case, (6) and (7) must hold for fixed inputs (Fä re, et. al, 1989).

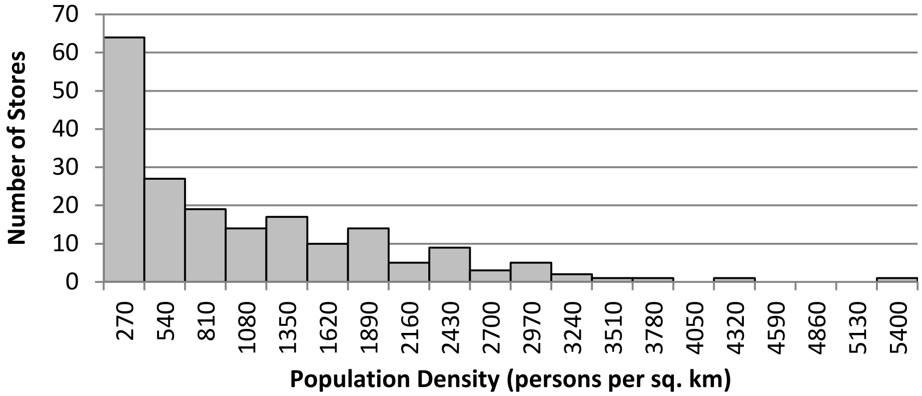
Appendix III: United States Mean Time Gardening per Capita in 2009



Appendix IV: Temperature/Precipitation Regions in the United States



Appendix V: Stores' Zip-Code Population Density



Appendix VI: Summary Statistics for Inputs and Outputs

Region	1	2	3	4	5	7	8
Total Observations (store/weeks)	623	1986	1139	1456	2912	1712	156
Inputs, mean and (standard deviation)							
Garden Time Per Capita (minutes per day)	13.2 (18.6)	13.1 (17.4)	17.5 (21.9)	9.5 (7)	10 (5.2)	7.3 (8.2)	7.7 (6.8)
Earnings Per Capita (USD per week)	704 (86)	954 (271)	834 (115)	759 (134)	870 (128)	807 (80)	758 (36)
Population Density (persons per sq km)	256.2 (404.9)	989.9 (824)	618.7 (470)	875.4 (927.2)	1146.1 (1129.5)	1005.7 (890.7)	772.1 (669.8)
Inputs, Inventory Units (% > 0)							
Vegetables	76%	72%	62%	100%	100%	100%	100%
Succulents	99%	97%	97%	98%	99%	98%	99%
Azaleas	39%	72%	78%	79%	94%	54%	79%
Annuals	79%	84%	78%	99%	99%	100%	96%
Cypress	71%	76%	94%	25%	27%	4%	0%
Outputs: Sales Units (% > 0)							
Vegetables	43%	48%	30%	96%	99%	93%	97%
Succulents	67%	74%	67%	87%	95%	90%	96%
Azaleas	20%	53%	56%	57%	80%	22%	44%
Annuals	64%	70%	57%	99%	100%	98%	100%
Cypress	52%	60%	84%	13%	15%	3%	0%

Appendix VII: Mean Subsample Size for Bootstrap

	Region 1	Region 2	Region 3	Region 4	Region 5	Region 7	Region 8
All	166	591	334	459	932	558	47
Qr 1	43	154	89	117	249	147	13
Qr 2	42	133	76	109	204	139	13
Qr 3	45	152	78	110	219	130	9
Qr 4	36	152	91	122	260	142	12

Appendix VIII: Derivation of Matlab Code

Here we present the derivation of the Matlab code. To begin, we have the standard directional DEA problem:

(A.5) $\phi(x^k) = \max_{z,\beta} \beta \;\; s.t.$

$$\sum_{\forall k} z^k y_m^k \geq y_m^k + \beta g_m, m = 1, \dots, M$$
$$\sum_{\forall k} z^k x_n^k \leq x_n^k + \beta g_n, n = 1, \dots, N$$
$$z^k \geq 0; k = 1, \dots, K; \sum_{k=1}^{K} z^k \leq 1;$$

We measure in the direction of the k^{th} observation's output vector, only, i.e. $g_m = y_{k,m}$ and $g_n = 0$. By rearranging equations the problem becomes:

(A.6) $\phi(x^k) = \max_{z,\beta} \beta \;\; s.t.$

$$-\sum_{\forall k} z^k y_m^k + y_m^k + \beta y_m^k \leq 0, m = 1, \dots, M$$
$$\sum_{\forall k} z^k x_n^k - x_n^k \leq 0, n = 1, \dots, N$$
$$z^k \geq 0; k = 1, \dots, K; \sum_{k=1}^{K} z^k \leq 1;$$

By inverting the problem to become a minimization problem we have

(A.7) $\phi(x^k) = \min_{z,\beta} -\beta \;\; s.t.$

$$-\sum_{\forall k} z^k y_m^k + y_m^k + \beta y_m^k \leq 0, m = 1, \dots, M$$
$$\sum_{\forall k} z^k x_n^k - x_n^k \leq 0, n = 1, \dots, N$$
$$z^k \geq 0; k = 1, \dots, K; \sum_{k=1}^{K} z^k \leq 1;$$

This can be expressed using matrix notation as:

(A.8) $\phi(x^k) = \min_{z,\beta} -\beta \;\; s.t.$

$$-YZ + Y^k + Y^k\beta \leq 0$$
$$ZX - X^k + \vec{0}\beta \leq 0$$
$$Z \geq 0; Z \cdot \vec{1} \leq 1;$$

Where Y is a m X k matrix, and X is a n X k matrix., Z is a k X 1 matrix, and Y^k is the kth column of the Y matrix.

And can be reduced to:

(A.9) $\phi(x^k) = \min_B f'B \;\; s.t.\;\; AB \leq 0; Z \geq 0; [1 \quad \cdots \quad 1 \quad 0 \quad 0] \cdot B = 1$

Where…

$$A = \begin{bmatrix} -Y & Y^k & Y^k \\ X & -X^k & \vec{0} \\ \vec{1} & 0 & 0 \end{bmatrix}$$ is a $(m+n+1) \times (k+2)$ matrix

$B = [Z \quad 1 \quad \beta]$ is a $(k+2) \times 1$ vector.

$f = [0 \quad ... \quad 0 \quad 0 \quad -1]$ is a $1 \times (k+2)$ vector.

All the setups above may be modified to find $\hat{\phi}(x^k)$ by redefining X to have rows $n = 2, \ldots, N$.

Appendix IX: FLT as Percentage of Revenue or Product Lost Due to Time at the Frontier:

$$RLT^k = Time\ Unconstrained\ Revenue - Time\ Constrained\ Revenue \;\Leftrightarrow$$

$$RLT^k = \sum_{m=1}^{M}\left[\left(\hat{D}_T^k\left(x_f, y; \vec{0}, y\right) + 1\right)y_m^k p_m^k\right] - \sum_{m=1}^{M}\left[\left(\vec{D}_T^k\left(x, y; \vec{0}, y\right) + 1\right)y_m^k p_m^k\right] \;\Leftrightarrow$$

$$RLT^k = \sum_{m=1}^{M}\left[\left(\hat{D}_T^k\left(x_f, y; \vec{0}, y\right) + 1\right)y_m^k p_m^k\right] - \sum_{m=1}^{M}\left[\left(\vec{D}_T^k\left(x, y; \vec{0}, y\right) + 1\right)y_m^k p_m^k\right] \;\Leftrightarrow$$

$$RLT^k = \left(\hat{D}_T^k\left(x_f, y; \vec{0}, y\right) + 1\right)\sum_{m=1}^{M} y_m^k p_m^k - \left(\vec{D}_T^k\left(x, y; \vec{0}, y\right) + 1\right)\sum_{m=1}^{M} y_m^k p_m^k \Leftrightarrow$$

$$RLT^k = \left[\hat{D}_T^k\left(x_f, \vec{0}; \vec{0}, y\right) - \vec{D}_T^k\left(x, \vec{0}; \vec{0}, y\right)\right]\sum_{m=1}^{M} y_m^k p_m^k \;\Leftrightarrow$$

$$RLT^k = FLT^k \sum_{m=1}^{M} y_m^k p_m^k \;\Leftrightarrow$$

$$FLT^k = \frac{RLT^k}{\sum_{m=1}^{M} y_m^k p_m^k}$$

Similarly, by defining Product Lost due to Time:

$$PLT^k = Time\ Unconstrained\ Product - Time\ Constrained\ Product \;\Leftrightarrow$$

$$PLT^k = \sum_{m=1}^{M}\left[\left(\hat{D}_T^k\left(x_f, y; \vec{0}, y\right) + 1\right)y_m^k\right] - \sum_{m=1}^{M}\left[\left(\vec{D}_T^k\left(x, y; \vec{0}, y\right) + 1\right)y_m^k\right] \;\Leftrightarrow$$

$$FLT^k = \frac{PLT^k}{\sum_{m=1}^{M} y_m^k}$$